

THE SPACE OF ANOSOV DIFFEOMORPHISMS

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ABSTRACT. We consider the space \mathcal{X}_L of Anosov diffeomorphisms homotopic to a fixed automorphism L of an infranilmanifold M . We show that if M is the 2-torus \mathbb{T}^2 then \mathcal{X}_L is homotopy equivalent to \mathbb{T}^2 . In contrast, if dimension of M is large enough, we show that \mathcal{X}_L is rich in homotopy and has infinitely many connected components.

1. INTRODUCTION

Let M be a smooth compact n -dimensional manifold that supports an Anosov diffeomorphism. Recall that a diffeomorphism $f: M \rightarrow M$ is called *Anosov* if there exist constants $\lambda \in (0, 1)$ and $C > 0$ along with a df -invariant splitting $TM = E^s \oplus E^u$ of the tangent bundle of M , such that for all $m \geq 0$

$$\begin{aligned} \|df^m v\| &\leq C\lambda^m \|v\|, \quad v \in E^s, \\ \|df^{-m} v\| &\leq C\lambda^m \|v\|, \quad v \in E^u. \end{aligned}$$

Currently the only known examples of Anosov diffeomorphisms are Anosov automorphisms of infranilmanifolds and diffeomorphisms conjugate to them. Furthermore, global structural stability of Franks and Manning [Fr70, M74] asserts that any Anosov diffeomorphism f of an infranilmanifold is conjugate to an Anosov automorphism L with the conjugacy being homotopic to identity. See, e.g., [KH95] for the background on Anosov diffeomorphisms.

In the light of the above discussion we fix an infranilmanifold M and an Anosov automorphism $L: M \rightarrow M$. We shall study the space \mathcal{X}_L of Anosov diffeomorphisms of M that are homotopic to L . In other words, an Anosov diffeomorphism f belongs to \mathcal{X}_L if and only if there exists a continuous path of maps $f_t: M \rightarrow M$ such that $f_0 = L$ and $f_1 = f$. If one has a smooth path of diffeomorphisms (rather than maps) connecting L and f then we say that f is *isotopic* to L . We equip \mathcal{X}_L with C^r -topology, $r = 1, 2, \dots, \infty$.

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Denote by $Diff_0(M)$ the group of diffeomorphisms of M that are homotopic to identity. Also denote by $Top_0(M)$ the group of homeomorphisms of M that are homotopic to identity. Equip $Top_0(M)$ with compact-open topology.

The group $Diff_0(M)$ acts on \mathcal{X}_L by conjugation. Assume for a moment that L has only one fixed point and $M = \mathbb{T}^n$. This guarantees, by [W70], uniqueness of the conjugacy given by global structural stability. That is, for every $f \in \mathcal{X}_L$ there exists unique $h \in Top_0(\mathbb{T}^n)$ such that $f = h \circ L \circ h^{-1}$. Therefore we have the following inclusions

$$Diff_0(\mathbb{T}^n) \hookrightarrow \mathcal{X}_L \hookrightarrow Top_0(\mathbb{T}^n) \quad (*)$$

with the composition $Diff_0(\mathbb{T}^n) \hookrightarrow Top_0(\mathbb{T}^n)$ being the natural inclusion. Therefore, one gets topological information about the space \mathcal{X}_L from that on $Diff_0(\mathbb{T}^n)$ and $Top_0(\mathbb{T}^n)$. We will make precise statements and arguments below which are valid for the general case; i.e., when M is perhaps not \mathbb{T}^n or when L has possibly more than one fixed point.

2. RESULTS

Our goal is to provide some information on homotopy type of the space of Anosov diffeomorphisms \mathcal{X}_L . We start by recalling the definition an Anosov automorphism.

An *infranilmanifold* is a double coset space $M \stackrel{\text{def}}{=} G \backslash N \rtimes G / \Gamma$, where N is a simply connected nilpotent Lie group, G is a finite group, and Γ is a torsion-free discrete cocompact subgroup of the semidirect product $N \rtimes G$. When G is trivial M is called *nilmanifold*. An automorphism $\tilde{L}: N \rightarrow N$ is called *hyperbolic* if the differential $D\tilde{L}: \mathfrak{n} \rightarrow \mathfrak{n}$ does not have eigenvalues of absolute value 1. If an affine map $L \stackrel{\text{def}}{=} v \cdot \tilde{L}$ commutes with Γ then it induces an affine diffeomorphism L on M . It is easy to show that if \tilde{L} is hyperbolic then L is Anosov. And in this case we call L an *Anosov automorphism*.

Theorem 1. *Let $L: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an Anosov automorphism of the 2-torus. Then the space \mathcal{X}_L of C^r , $r > 1$, Anosov diffeomorphisms homotopic to L is homotopy equivalent to \mathbb{T}^2 .*

The proof relies on some standard results and techniques from hyperbolic dynamics. The outline of the proof is given in Appendix A.

Next we collect information about homotopy of $Diff_0(M)$ for higher dimensional M . Below \mathbb{Z}_p^∞ stands for the direct sum of countably many copies of $\mathbb{Z}_p \stackrel{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$.

Proposition 2. *If $n \geq 10$, then*

$$\pi_0(\text{Diff}_0(\mathbb{T}^n)) \simeq \mathbb{Z}_2^\infty \oplus \binom{n}{2} \mathbb{Z}_2 \oplus \sum_{i=0}^n \binom{n}{i} \Gamma_{i+1},$$

where Γ_i , $i = 0, \dots, n$, are the finite abelian groups of Kervaire-Milnor “exotic” spheres. Moreover, \mathbb{Z}_2^∞ maps monomorphically into $\pi_0(\text{Top}_0(\mathbb{T}^n))$ via the map induced by the inclusion $\text{Diff}_0(\mathbb{T}^n) \hookrightarrow \text{Top}_0(\mathbb{T}^n)$.

Proof. This result is contained in Theorem 4.1 of [H78] and Theorem 2.5 of [HS76] with one caveat. The proofs of both of these theorems depended strongly on a formula given in [HW73] and [H73]; cf. Theorem 3.1 of [H78]. Igusa found that this formula and its proof were seriously flawed, and he corrected this formula in Theorem 8.a.2 of [I84]. Using Igusa’s formula, the two proofs of Proposition 2 mentioned above are valid with minor modifications. \square

Proposition 3. *Let p be a prime number different from 2 and k be an integer satisfying $2p - 4 \leq k < \frac{n-7}{3}$. Then $\pi_k(\text{Diff}_0(\mathbb{T}^n))$ contains a subgroup S such that*

1. $S \simeq \mathbb{Z}_p^\infty$ and
2. S maps monomorphically into $\pi_k(\text{Top}_0(\mathbb{T}^n))$ via the map induced by the inclusion $\text{Diff}_0(\mathbb{T}^n) \hookrightarrow \text{Top}_0(\mathbb{T}^n)$.

We postpone the proof of the above proposition to Appendix B.

Proposition 4. *If M is an infranilmanifold of dimension $n \geq 10$ then*

$$\mathbb{Z}_2^\infty < \pi_0(\text{Diff}_0(M)).$$

Moreover, \mathbb{Z}_2^∞ maps monomorphically into $\pi_0(\text{Top}_0(M))$ via the map induced by the inclusion $\text{Diff}_0(M) \hookrightarrow \text{Top}_0(M)$.

Proof. This result follows from a slightly augmented form of Proposition 2.2(A) in [HS76] with the same caveat made in the proof of our Proposition 2. Since M is an infranilmanifold, $\pi_1(M)$ contains a normal nilpotent subgroup N with a finite quotient group $\pi_1(M)/N$. Now note that the center $\mathcal{Z}(N)$ of N is a finitely generated, infinite abelian group. Hence $\mathcal{Z}(\pi_1(M)) = \pi_1(\text{Aut}(M))$ is also finitely generated (but perhaps not infinite); thus verifying one of the hypotheses of Proposition 2.2(A). And since the $\pi_1(M)$ conjugacy class of any element in $\mathcal{Z}(N)$ is finite, $\pi_1(M)$ contains an infinite number of distinct conjugacy classes; therefore

$$Wh_1(\pi_1(M); \mathbb{Z}_2) = \mathbb{Z}_2^\infty.$$

Then

$$Wh_1(\pi_1(M); \mathbb{Z}_2) / \{c + \varepsilon \bar{c}\} = H_0(\mathbb{Z}_2; \mathbb{Z}_2^\infty) \simeq \mathbb{Z}_2^\infty$$

by the simple algebraic argument given on page 287 of [F02]. That is, we don't need to know whether " $\pi_1(M)$ contains infinitely many conjugacy classes distinct from their inverse classes" as hypothesized in Proposition 2.2 of [HS76] to complete the argument given in that paper which produces a subgroup \mathbb{Z}_2^∞ of $\pi_0(\text{Diff}_0(M))$. (The diffeomorphisms representing the elements of \mathbb{Z}_2^∞ are all homotopic to id_M since they are constructed to be pseudo-isotopic to id_M .) Since Proposition 2.2 is also true in the topological category (cf. footnote (i) on page 401 of [HS76]), this subgroup \mathbb{Z}_2^∞ maps monomorphically into $\pi_0(\text{Top}_0(M))$. \square

Proposition 5. *Let M be an n -dimensional infranilmanifold and p be a prime number different from 2. Assume that the first Betti number of M is non-zero, i.e., $H_1(M, \mathbb{Q}) \neq 0$ and that $n > \max\{9, 6p - 5\}$. Then there exists a subgroup S of $\pi_{2p-4}(\text{Diff}_0(M))$ such that*

1. $S \simeq \mathbb{Z}_p^\infty$ and
2. S maps monomorphically into $\pi_{2p-4}(\text{Top}_0(M))$ via the map induced by the inclusion $\text{Diff}_0(M) \hookrightarrow \text{Top}_0(M)$.

Remark. The first Betti number of any nilmanifold is different from zero.

We postpone the proof of the above proposition to Appendix B.

Theorem 6. *Let M be an n -dimensional infranilmanifold, $L: M \rightarrow M$ be an Anosov automorphism and \mathcal{X}_L be the space of Anosov diffeomorphisms homotopic to L then the following is true*

1. If $M = \mathbb{T}^n$ and $n \geq 10$, then \mathcal{X}_L has infinitely many connected components.
2. If $M = \mathbb{T}^n$, p is a prime number different from 2 and k is an integer satisfying $2p - 4 \leq k < \frac{n-7}{3}$, then

$$\mathbb{Z}_p^\infty < \pi_k(\mathcal{X}_L).$$

3. Let M be an infranilmanifold of dimension $n \geq 10$, then \mathcal{X}_L has infinitely many connected components.
4. If p is a prime number different from 2 and M is an infranilmanifold of dimension $n > \max\{9, 6p - 5\}$ with a non-zero first Betti number then

$$\mathbb{Z}_p^\infty < \pi_{2p-4}(\mathcal{X}_L).$$

Proof. We prove the third statement only. The proofs of the other statements are easier and follow the same lines. The proof will rely on the following statement whose proof we postpone to Appendix C.

Proposition 7. *Let M be an infranilmanifold. Assume that a homeomorphism $h: M \rightarrow M$ is homotopic to identity and commutes with an Anosov automorphism $L: M \rightarrow M$. Then h is isotopic to id_M .*

Take $h_1, h_2 \in Diff_0(M)$ that represent different elements $[h_1], [h_2]$ of $\mathbb{Z}_2^\infty < \pi_0(Diff_0(M))$. Let $L_1 = h_1 \circ L \circ h_1^{-1}$ and $L_2 = h_2 \circ L \circ h_2^{-1}$. Assume that there is a path of Anosov diffeomorphisms L_t , $t \in [1, 2]$, connecting L_1 and L_2 . By structural stability and local uniqueness we get a continuous path $\{\tilde{h}_t, t \in [1, 2]\}$ in $Top_0(M)$ such that $\tilde{h}_1 = h_1$ and $L_t = \tilde{h}_t \circ L \circ \tilde{h}_t^{-1}$. Hence we get

$$h_2 \circ L \circ h_2^{-1} = \tilde{h}_2 \circ L \circ \tilde{h}_2^{-1}.$$

The homeomorphism $\tilde{h}_2^{-1} \circ h_2$ commutes with L . Hence Proposition 7 implies that $[\tilde{h}_2^{-1} \circ h_2] = [id_M]$. Therefore

$$[h_2] = [\tilde{h}_2] = [h_1] \text{ in } \pi_0(Top_0(M)),$$

which gives us a contradiction. We conclude that L_1 and L_2 represent different connected components of \mathcal{X}_L . \square

Remarks.

1. It is not clear whether or not there are other connected components of \mathcal{X}_L that we are not detecting. Question: for which $h \in Diff_0(M)$, does the connected component of $h \circ L$ in $Diff(M)$ contain an Anosov diffeomorphism?
2. By Moser's homotopy trick the space of $Diff_0(M)^{vol}$ consisting of all volume preserving diffeomorphisms homotopic to id_M is a deformation retraction of $Diff_0(M)$. Hence, we have analogous results for the space of volume preserving Anosov diffeomorphisms \mathcal{X}_L^{vol} .
3. See page 10 of [H78] for a conjectural geometric description for representatives of non-zero elements in $\mathbb{Z}_2^\infty < \pi_0 Diff_0(M)$ of Propositions 2 and 4.

APPENDIX A. SKETCH OF THE PROOF OF THEOREM 1

Convention. When we say that an object is C^{1+} we mean that it is C^1 and the first derivative is Hölder continuous with some positive exponent.

We start by introducing some notation and recalling some definitions. Given an Anosov diffeomorphism $g: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ we denote by $W^s(g)$ and $W^u(g)$ the stable and unstable foliations of g . We assume that a background Riemannian metric a is fixed. The logarithms of stable and unstable jacobians of g will be denoted by $\varphi^s(g)$ and $\varphi^u(g)$.

Two Hölder continuous functions $\varphi_1, \varphi_2: \mathbb{T}^2 \rightarrow \mathbb{R}$ are called *cohomologous up to an additive constant over g* if there exist a constant C and a Hölder continuous function $u: \mathbb{T}^2 \rightarrow \mathbb{R}$ such that $\varphi_1 = \varphi_2 + C + u \circ g - u$. In this case we write $\langle \varphi_1 \rangle = \langle \varphi_2 \rangle$. We remark that even though $\varphi^s(g)$ and $\varphi^u(g)$ depend on a , the cohomology classes $\langle \varphi^s(g) \rangle$ and $\langle \varphi^u(g) \rangle$ are independent of the choice of a .

Let p_0 be a fixed point of L , $L(p_0) = p_0$. For every point $p \in \mathbb{T}^2$ consider the translation $t_p: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ that takes p_0 into p . Then $L_p \stackrel{\text{def}}{=} t_p \circ L \circ t_p^{-1}$ is an Anosov automorphism that fixes point p .

Recall that the space of all C^r diffeomorphisms of \mathbb{T}^2 — $\text{Diff}(\mathbb{T}^2)$ — is a separable infinite dimensional manifold modeled on the Banach space or the Fréchet space of C^r vector fields when $r < \infty$ or $r = \infty$, respectively. Hence $\text{Diff}(\mathbb{T}^2)$ is a separable absolute neighborhood retract. Since \mathcal{X}_L is an open subset of $\text{Diff}(\mathbb{T}^2)$, we also have that \mathcal{X}_L is a separable absolute neighborhood retract. By a result of W.H.C. Whitehead [P66], every absolute neighborhood retract has the homotopy type of a CW complex. Therefore \mathcal{X}_L has homotopy type of a CW complex.

Our goal is to show that \mathcal{X}_L is homotopy equivalent to the 2-torus \mathbb{T}^2 which we identify with $\{L_p, p \in \mathbb{T}^2\} \subset \mathcal{X}_L$. (Note that the map $\mathbb{T}^2 \ni p \rightarrow L_p \in \mathbb{T}^2$ is a finite covering.) Let \mathbb{D}^k be a disk of dimension k and let $\alpha: \mathbb{D}^k \rightarrow \mathcal{X}_L$ be a continuous map which sends the boundary of the disk to L . We shall show that α can be homotoped to $\hat{\alpha}: \mathbb{D}^k \rightarrow \mathbb{T}^2$. This implies that the inclusion $\mathbb{T}^2 \hookrightarrow \mathcal{X}_L$ induces an epimorphism on homotopy groups. Therefore, since \mathbb{T}^2 is aspherical, $\pi_k(\mathbb{T}^2) \rightarrow \pi_k(\mathcal{X}_L)$ is a trivial isomorphism for $k \geq 2$. The homomorphism $\pi_1(\mathbb{T}^2) \rightarrow \pi_1(\mathcal{X}_L)$ is monic and, hence, is an isomorphism as well. This can be seen from structural stability and the fact that $\pi_1(\mathbb{T}^2) \rightarrow \pi_1(\text{Top}_0(\mathbb{T}^2))$ is monic. Now, since \mathcal{X}_L has homotopy type of a CW complex, Theorem 1 will follow from J.H.C. Whitehead's Theorem.

Structural stability gives the continuation $\xi: \mathbb{D}^k \rightarrow \text{Top}_0(\mathbb{T}^2)$ to the interior of the disk of the conjugacy to the linear model:

$$\alpha(\cdot) \circ \xi(\cdot) = \xi(\cdot) \circ L.$$

If $k \geq 2$ we can assume that $\xi(x) = id_{\mathbb{T}^2}$ for $x \in \partial \mathbb{D}^k$. Then $fix(\cdot) \stackrel{\text{def}}{=} \xi(\cdot)(p_0)$ defines the continuation of the fixed point p_0 to the interior of the disk, $\alpha(\cdot)(fix(\cdot)) = fix(\cdot)$. If $k = 1$ then ξ is $id_{\mathbb{T}^2}$ at one endpoint and a possibly non-trivial translation at the other endpoint. In this case fix is defined as before and gives a path that connects p_0 with a fixed point of L . We shall explain how to construct a homotopy that connects α to $\hat{\alpha} \stackrel{\text{def}}{=} L_{fix(\cdot)}$.

Take $x \in \mathbb{D}^k$ and let $f \stackrel{\text{def}}{=} \alpha(x)$. Let $p \stackrel{\text{def}}{=} \text{fix}(x)$ be the fixed point of f . Next we construct a path of diffeomorphisms that connects f and f_p , where f_p is a C^{1+} Anosov diffeomorphism C^{1+} conjugate to L_p . The path will consist of Anosov diffeomorphisms of regularity C^{1+} that fix p .

Choose a simple closed curve T which is transverse to $W^u(f)$ and passes through p . Transversal T cuts the leaves of the unstable foliation $W^u(f)$ into oriented arcs $[y, e(y)]$ parameterized by $y \in T$.

Given a Hölder continuous potential $\varphi: \mathbb{T}^2 \rightarrow \mathbb{R}$ where exists a unique system of measures $\{\mu_y^\varphi, y \in T\}$, satisfying the following properties.

1. $\mu_y^\varphi, y \in T$, are finite measures supported on $[y, e(y)]$;
2. $\frac{d(f^* \mu_{\bar{y}}^\varphi)}{d\mu_y^\varphi} = e^{\varphi(z) - P(\varphi)}$, where \bar{y} is the base point of the arc that contains $f(z)$ and $P(\varphi)$ is the pressure of φ ;
3. the system $\{\mu_y^\varphi\}$ satisfies certain absolute continuity property with respect to the stable foliation $W^s(f)$.

Measures μ_y^φ are equivalent to the conditional measures on $[y, e(y)]$ of the equilibrium state of φ . Notice that if $\varphi = \varphi^u(f)$ then μ_y^φ are absolutely continuous measures induced by the Riemannian metric a . For more details about the system $\{\mu_y^\varphi\}$ and the proof of existence and uniqueness, see, e.g., [L00].

Consider the path of potentials $\varphi_t \stackrel{\text{def}}{=} (1 - 2t)\varphi^u(f)$, $t \in [0, 1/2]$. Corresponding system of measures $\mu_y^{\varphi_t}$ depends continuously on t (see, e.g., [C92]).

Now we can define a C^{1+} path of Anosov diffeomorphisms whose logarithmic unstable jacobians are cohomologous up to a constant to φ_t . This is done in the following way.

Consider the functions $\eta_t: T \rightarrow \mathbb{R}$ given by $\eta_t(y) = \mu_y^{\varphi_t}([y, e(y)])$. Choose a continuous family of Riemannian metrics a_t , $t \in [0, 1/2]$, such that $a_0 = a$ and the induced lengths $l_y^t([y, e(y)])$ of the arcs $[y, e(y)]$ in the metric a_t equal to $\eta_t(y)$. Consider the family of homeomorphisms $h_t: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ that preserve the partition by the arcs $[y, e(y)]$, $y \in T$, and satisfy the following relation

$$\mu_y^{\varphi_t}([y, z]) = l_y^t([y, h_t(z)]), z \in [y, e(y)].$$

Clearly, the family of homeomorphisms h_t is uniquely determined by these properties. Define

$$f_t \stackrel{\text{def}}{=} h_t \circ f \circ h_t^{-1}, t \in [0, 1/2].$$

Then $f_0 = f$ and it is easy to check that f_t , $t \in [0, 1/2]$, are C^{1+} Anosov diffeomorphisms with $\langle \varphi^u(f_t) \rangle = \langle \varphi_t \circ h_t^{-1} \rangle$ over f_t . Because the stable foliation $W^s(f)$ is C^{1+} , it follows that $\langle \varphi^s(f_t) \rangle = \langle \varphi^s(f) \circ h_t^{-1} \rangle$, $t \in [0, 1/2]$.

Now we switch the roles of the stable and the unstable foliations and apply the same construction to $f_{1/2}$ to get a path f_t , $t \in [1/2, 1]$, connecting $f_{1/2}$ to f_1 . Then $\langle \varphi^s(f_1) \rangle = \langle 0 \rangle$, $\langle \varphi^s(f_1) \rangle = \langle \varphi^s(f_{1/2}) \rangle = \langle 0 \rangle$ and it follows that $f_p \stackrel{\text{def}}{=} f_1$ is C^{1+} conjugate to L_p [dlL92].

It is routine to check that the construction outlined above can be carried out simultaneously for all $\alpha(x)$, $x \in \mathbb{D}^k$, and that the resulting homotopy can be made to be constant on $\partial \mathbb{D}^k$. The choices of transversals $T = T(x)$ and families of Riemannian metrics $a_t = a_t(x)$ must be made continuously in x to make sure that a (continuous) homotopy of α is produced. This homotopy connects α and $\tilde{\alpha}: D \rightarrow \text{Diff}^{1+}(\mathbb{T}^2)$ whose image lies in C^{1+} conjugacy class of L . Using standard smoothing methods we can C^1 approximate our homotopy by another one that takes values in \mathcal{X}_L and connects α to $\tilde{\alpha}: D \rightarrow \mathcal{X}_L$.

The map $\tilde{\alpha}$ can be C^1 approximated by a map whose image lies in the C^r conjugacy class of L simply by approximating C^{1+} conjugacy with a C^r conjugacy. This map is C^1 close to $\tilde{\alpha}$ and hence, by performing a short homotopy if needed, we can assume that the map $\tilde{\alpha}$ is C^r conjugate to L .

Finally, map $\tilde{\alpha}$ can be homotoped to the map $\hat{\alpha}: \mathbb{D} \rightarrow \mathbb{T}^2$ by homotoping corresponding map $h: \mathbb{D}^k \rightarrow \text{Diff}_0(M)$, $h(\cdot) \circ L \circ h^{-1}(\cdot) = \tilde{\alpha}(\cdot)$, in the space of C^r conjugacies to a map consisting of the translations $t: \mathbb{D}^k \rightarrow \text{Diff}_0(M)$ given by $t(x) = t_{\text{fix}(\tilde{\alpha}(x))} = t_{\text{fix}(\alpha(x))}$. The latter is possible due to a result of Earle and Eells [EE69] who showed that $\mathbb{T}^2 = \{t_p, p \in \mathbb{T}^2\}$ is a deformation retraction of the space of smooth conjugacies $\text{Diff}_0(\mathbb{T}^2)$.

APPENDIX B. PROOFS OF PROPOSITIONS 2 AND 4

Remark. We write $k \ll n$ for $\max\{3k+7, 9\} < n$; in particular, $2p-4 \ll n$ if and only if $\max\{9, 6p-5\} < n$.

Proof of Proposition 5. Consider the following commutative diagram

$$\begin{array}{ccccc}
 P^s(T) & \longrightarrow & P^s(M) & \xrightarrow{t} & \text{Diff}_0(M) \\
 \downarrow & & \downarrow & & \downarrow \\
 P(T) & \longrightarrow & P(M) & \xrightarrow{t} & \text{Top}_0(M)
 \end{array} \tag{*}$$

where T is a closed tubular neighborhood of a smooth simple closed curve α in M such that the homology class represented by α generates an infinite cyclic direct summand of $H_1(M)$.

Remark. If M is orientable then $T = S^1 \times \mathbb{D}^{n-1}$. In general it is the mapping torus of a self-diffeomorphism of \mathbb{D}^{n-1} .

In this diagram $P^s(\cdot)$ and $P(\cdot)$ are the smooth and topological pseudo-isotopy functors, respectively. Recall that a topological (smooth) pseudo-isotopy of a compact manifold M is a homeomorphism (diffeomorphism)

$$f: M \times [0, 1] \rightarrow M \times [0, 1]$$

such that $f(x) = x$ for all $x \in M \times 0 \cup \partial M \times [0, 1]$. Then $P(M)$ ($P^s(M)$) is the topological space consisting of all such homeomorphisms (diffeomorphisms), respectively. Since $T \subset M$ is a codimension 0 submanifold, a pseudo-isotopy f of T canonically induces a pseudo-isotopy F of M by setting $F(x) = f(x)$, when $x \in T \times [0, 1]$, and $F(x) = x$ otherwise.

Note that the pseudo-isotopy f must map $M \times 1$ into itself. Hence, after identifying M with $M \times 1$ in the obvious way, the restriction of f to $M \times 1$ determines an element in $Top_0(M)$ or $Diff_0(M)$ depending on whether $f \in P(M)$ or $P^s(M)$. This restriction gives the maps t (standing for top) in diagram (\star) .

Now Proposition 5 is clearly implied by the following Assertion.

Assertion. *Let M be an n -dimensional infranilmanifold and p be a prime number different from 2. Assume that the first Betti number of M is non-zero and that $n > \max\{9, 6p - 5\}$. Then there exists a subgroup S of $\pi_{2p-4}(P^s(T))$ such that*

1. $S \simeq \mathbb{Z}_p^\infty$ and
2. S maps into $\pi_{2p-4}(Top_0(M))$ with a finite kernel via the homomorphism which is functorially induced by the maps in the commutative diagram (\star) .

This Assertion will be proven by concatenating several facts which we now list.

Fact 1. *The kernel of the homomorphism $\pi_k P^s(T) \rightarrow \pi_k P(T)$ is a finitely generated abelian group provided $k \ll n$.*

This fact follows from Corollary 4.2 in [FO10].

Fact 2. *Denote the inclusion map $T \subset M$ by σ . Then the induced homomorphism $\pi_k P(\sigma): \pi_k P(T) \rightarrow \pi_k P(M)$ is monic provided $k \ll n$.*

Fact 2 is proven as follows. Since the class of α in $H_1(M)$ generates an infinite cyclic direct summand, there clearly exists a continuous map $\gamma: M \rightarrow S^1$ such that the composition

$$S^1 \xrightarrow{\alpha} T \xrightarrow{\sigma} M \xrightarrow{\gamma} S^1$$

is homotopic to id_{S^1} . Let $\mathcal{P}(\cdot)$ denote the stable topological pseudo-isotopy functor. Applying $\mathcal{P}(\cdot)$ to the above composition yields that

$$\pi_k \mathcal{P}(\sigma): \pi_k \mathcal{P}(T) \rightarrow \pi_k \mathcal{P}(M)$$

is monic since $\mathcal{P}(\cdot)$ is a homotopy functor; cf. [H78]. Therefore Igusa's stability result [I88] completes the proof of Fact 2.

There is an involution “ $-$ ” defined on $P(M)$ which is essentially determined by “turning a pseudo-isotopy upside down.” See pages 6 and 18 of [H78] for a precise definition. (Also see page 298 of [FO10].) Since σ commutes with “ $-$ ”, it induces a homomorphism

$$H_0(\mathbb{Z}_2, \pi_k P(T)) \rightarrow H_0(\mathbb{Z}_2, \pi_k P(M)).$$

Fact 3. *This homomorphism is monic provided $k \ll n$.*

Fact 3 is proven by an argument similar to that given for Fact 2

Fact 4. *If $k \ll n$, then $\pi_k P(t): \pi_k P(M) \rightarrow \pi_k Top_0(M)$ factors through a homomorphism*

$$\varphi: H_0(\mathbb{Z}_2; \pi_k P(M)) \rightarrow \pi_k Top_0(M),$$

whose kernel contains only elements of order a power of 2.

Fact 4 follows from Hatcher's spectral sequence (see pages 6 and 7 of [H78]) by using that topological rigidity holds for all infranilmanifolds (proven in [FH83]). The argument is a straightforward extension of the one proving Corollary 5 in Section 5 of [F02]

Fact 5. *There is a subgroup S of $\pi_k P^s(T)$, where $k = 2p - 4 \ll n$, satisfying*

1. $S \simeq \mathbb{Z}_p^\infty$ and
2. both $x \mapsto x + \bar{x}$ and $x \mapsto x - \bar{x}$ are monomorphisms of S into $\pi_k P^s(T)$.

When T is orientable, i.e., $T = S^1 \times \mathbb{D}^{n-1}$, then Fact 5 follows from Proposition 4.6 of [FO10] — which is the analogous result valid for $\pi_k \mathcal{P}^s(S^1)$ — by again using Igusa's stability theorem [I88]. We note that Proposition 4.6 depended on important calculations of $\pi_k \mathcal{P}^s(S^1)$ which can be found in [GKM08].

In the non-orientable case, $T = \mathcal{M} \times \mathbb{D}^{n-2}$ where \mathcal{M} denotes the Möbius band and we argue as follows. Since $S^1 \times \mathbb{D}^1$ is a collar neighborhood of the boundary $\partial \mathcal{M}$, we can also identify $S^1 \times \mathbb{D}^{n-1}$ with a

collar neighborhood of ∂T . There are two natural maps $S^1 \times \mathbb{D}^{n-1} \rightarrow T$; namely, the inclusion map ω of the collar neighborhood and the (2-sheeted) orientation covering map $q: S^1 \times \mathbb{D}^{n-1} \rightarrow T$ which induces a transfer map

$$q^*: P^s(T) \rightarrow P^s(S^1 \times \mathbb{D}^{n-1}).$$

And let

$$\omega_*: P^s(S^1 \times \mathbb{D}^{n-1}) \rightarrow P^s(T)$$

denote the natural (induction) map previously denoted by $P^s(\omega)$. A pleasant exercise shows that

$$\pi_k(q^* \circ \omega_*): \pi_k(S^1 \times \mathbb{D}^{n-1}) \rightarrow \pi_k(S^1 \times \mathbb{D}^{n-1})$$

is multiplication by 2. Hence the kernel of

$$\pi_k(\omega_*): \pi_k(S^1 \times \mathbb{D}^{n-1}) \rightarrow \pi_k(T)$$

contains only 2-torsion. Denote $\pi_k(\omega_*)$ by Ω and let S' be a subgroup of $\pi_k(P^s(S^1 \times \mathbb{D}^{n-1}))$ satisfying properties 1 and 2 of Fact 5. Then $S \stackrel{\text{def}}{=} \Omega(S')$ is a subgroup of $\pi_k P^s(T)$ which clearly satisfies property 1 since $p \neq 2$. To see that property 2 is also satisfied consider the homomorphism $\Phi: S' \rightarrow \pi_k P^s(T)$ defined by

$$\Phi(y) \stackrel{\text{def}}{=} \Omega(y + \bar{y}),$$

and observe that Φ is monic since the homomorphism

$$y \mapsto y + \bar{y}, \quad y \in S'$$

is monic and its image contains only p -torsion ($p \neq 2$). But

$$\Phi(y) = \Omega(y) + \overline{\Omega(y)} = x + \bar{x},$$

where $x = \Omega(y)$ is an arbitrary element of S . Consequently the homomorphism $x \mapsto x + \bar{x}$, $x \in S$ is also monic. A completely analogous argument shows that

$$x \mapsto x - \bar{x}, \quad x \in S$$

is monic completing the verification of property 2.

We now string together these facts to prove the Assertion. The group S of the Assertion is the one given in Fact 5. Applying the functor $\pi_k(\cdot)$

to diagram (\star) yields the following commutative diagram

$$\begin{array}{ccccc}
 S \subset \pi_k P^s(T) & \longrightarrow & \pi_k P^s(M) & \longrightarrow & \pi_k \text{Diff}_0(M) \\
 \eta \downarrow & & \downarrow \text{H} & & \downarrow \\
 \pi_k P(T) & \longrightarrow & \pi_k P(M) & \xrightarrow{\pi_k(P(t))} & \pi_k \text{Top}_0(M) \\
 \psi \downarrow & & \downarrow \Psi & \nearrow \varphi & \\
 H_0(\mathbb{Z}_2; \pi_k P(T)) & \xrightarrow{\varkappa} & H_0(\mathbb{Z}_2; \pi_k P(M)) & &
 \end{array} \quad (\star\star)$$

The triangle in diagram $(\star\star)$ is the factorization of $\pi_k(P(t))$ given in Fact 4. And the bottom vertical maps are the quotient homomorphisms

$$\pi_k P(X) \rightarrow \pi_k P(X) / \{x - \bar{x} : x \in \pi_k P(X)\},$$

where $X = T$ and M , respectively. Also we denote by η the homomorphism studied in Fact 1, and by \varkappa the monomorphism in Fact 3.

Combining Facts 1 and 5, we see that the kernel of $x \mapsto \eta(x + \bar{x})$, $x \in S$, is a finite group. Consequently, the kernel of

$$\psi \circ \eta : S \rightarrow H_0(\mathbb{Z}_2; \pi_k P(T))$$

is also a finite group. And since $\psi \circ \eta(S)$ is a p -torsion group, where $p \neq 2$, we see that

$$\varphi \circ \varkappa : \text{image}(\psi \circ \eta) \rightarrow \pi_k \text{Top}_0(M)$$

is monic by Facts 3 and 4. Therefore the composition

$$\varphi \circ \varkappa \circ \psi \circ \eta : S \rightarrow \pi_k \text{Top}_0(M)$$

has finite kernel. But this is the homomorphism of part 2 of the Assertion. \square

Proof of Proposition 3. Let $X \mapsto A(X)$ denote Waldhausen's algebraic K -theory of spaces functor defined in [W78]. We start with the following result.

Lemma 8. *For every prime $p \neq 2$ and every integer $k \in [2p - 4, (2p - 4) + n - 1]$, $\pi_k A(\mathbb{T}^n)$ contains a subgroup \mathbb{Z}_p^∞ such that the following two group endomorphisms*

$$x \mapsto x + \bar{x} \quad \text{and} \quad x \mapsto x - \bar{x}$$

are both monic when restricted to \mathbb{Z}_p^∞ .

Proof. We verify this by induction on n . For $n = 1$ it was verified in the proof of Proposition 4.6 from [FO10]. Now assume that Lemma 8 is true for n , we proceed to verify it for $n + 1$. Since $\mathbb{T}^{n+1} = \mathbb{T}^n \times S^1$, $\pi_k A(\mathbb{T}^{n+1})$ contains as subgroups both $\pi_k A(\mathbb{T}^n)$ and $\pi_{k-1} A(\mathbb{T}^n)$ in an involution consistent way; cf. [HKVWW02]. \square

Since Waldhausen proved in [W78] that the kernel of $\pi_k A(X) \rightarrow \pi_k P^s(X)$ is finitely generated, Igusa's stability theorem [I88] yields the following variant of Lemma 8. \square

Lemma 8'. *Let p be a prime number different from 2 and k be an integer such that $2p-4 \leq k < \frac{n-7}{3}$. Then $\pi_k P^s(\mathbb{T}^n)$ contains a subgroup \mathbb{Z}_p^∞ such that the following two group endomorphisms*

$$x \mapsto x + \bar{x} \quad \text{and} \quad x \mapsto x - \bar{x}$$

are both monic when restricted to \mathbb{Z}_p^∞ .

Now we follow the pattern used to prove Proposition 5 except that the argument is now simpler since the first column in both diagrams (\star) and $(\star\star)$ can be omitted. It clearly suffices to show that the subgroup \mathbb{Z}_p^∞ of $\pi_k P^s(\mathbb{T}^n)$, given by Lemma 8' maps with finite kernel into $\pi_k \text{Top}_0(\mathbb{T}^n)$ via composite homomorphism $\varphi \circ \Psi \circ H$. Since the kernel of

$$H: \pi_k P^s(\mathbb{T}^n) \rightarrow \pi_k P(M)$$

is finitely generated by Corollary 4.2 of [FO10], the kernel of $H|_{\mathbb{Z}_p^\infty}$ is finite. Now arguing as in the proof of Proposition 5, we see that the kernel of $\Psi \circ H: \mathbb{Z}_p^\infty \rightarrow H_0(\mathbb{Z}_2; \pi_k P(\mathbb{T}^n))$ is also finite. (Here we crucially use the fact from Lemma 8' that $x \mapsto x \pm \bar{x}$ is monic for $x \in \mathbb{Z}_p$.) Finally, we conclude from Fact 4 that

$$\varphi \circ (\Psi \circ H): \mathbb{Z}_p \rightarrow \pi_k \text{Top}_0(\mathbb{T}^n)$$

has finite kernel since $p \neq 2$.

APPENDIX C. PROOF OF PROPOSITION 7

The universal cover of the infranilmanifold M is a simply connected nilpotent Lie group N . The fundamental group $\pi_1(M)$ acts freely on the right by affine diffeomorphisms. Group N acts on itself by left translations. Let

$$\Gamma \stackrel{\text{def}}{=} \pi_1(M) \cap N.$$

It is well known that

$$\widehat{M} \stackrel{\text{def}}{=} N/\Gamma$$

is a compact nilmanifold and

$$G \stackrel{\text{def}}{=} \pi_1(M)/\Gamma \subset \mathbb{A}$$

is finite subgroup of *the group \mathbb{A} of all affine diffeomorphisms of \widehat{M}* . Group G acts freely on \widehat{M} ; the orbit space of this action is the infranilmanifold M .

Fact 1. *The centralizer $N^{\pi_1 M} \stackrel{\text{def}}{=} \{x \in N : axa^{-1} = x \text{ for all } a \in \pi_1(M)\}$ is path connected.*

Proof. Let $x \in N^{\pi_1 M}$. Since N is simply connected nilpotent Lie group, Mal'cev's work [M49] yields a 1-parameter subgroup $\alpha: \mathbb{R} \rightarrow N$ such that $\alpha(1) = x$. Let $a \in \pi_1(M)$, then

$$\beta(s) \stackrel{\text{def}}{=} a\alpha(s)a^{-1}$$

is also a 1-parameter subgroup such that $\beta(1) = a\alpha(1)a^{-1} = axa^{-1} = x$. Hence $\alpha(s) = \beta(s)$ for all $s \in \mathbb{R}$, again by Mal'cev's work. Therefore $\alpha(s) \in N^{\pi_1 M}$ for all $s \in \mathbb{R}$, yielding that $N^{\pi_1 M}$ is connected. \square

Group N also acts on \widehat{M} by left translations. In this way

$$N \rightarrow \mathbb{A} \subset \text{Diff}(\widehat{M})$$

and its image, denoted by \mathcal{N} , is a Lie group called *the group of translations of \widehat{M}* . Since the kernel of $N \rightarrow \mathbb{A}$ is $\mathcal{Z}(N) \cap \Gamma$ we have that

$$\mathcal{N} = N / \mathcal{Z}(N) \cap \Gamma.$$

Here $\mathcal{Z}(N)$ stands for the center of N .

Let $\mathbb{N} = \mathbb{N}(\mathcal{N}, G)$ denote the normalizer of G inside \mathcal{N} .

Fact 2. $\mathbb{N} = \mathcal{N}^G \stackrel{\text{def}}{=} \{x \in \mathcal{N} : gxg^{-1} = x \text{ for all } g \in G\}$.

Proof. Clearly $\mathcal{N}^G \subseteq \mathbb{N}$. Since \mathcal{N} is a normal subgroup of \mathbb{A}

$$[x, g] \in \mathcal{N} \cap G = 1$$

for all $x \in \mathbb{N}$ and $g \in G$. Therefore we also have $\mathbb{N} \subseteq \mathcal{N}^G$. \square

Note that the action by left translations of \mathbb{N} on \widehat{M} descends to an action by “translations” on M . Let t such a (left) translation on M that is homotopic to id_M and let $\widehat{t}: \widehat{M} \rightarrow \widehat{M}$ be a lift of t which is homotopic to $id_{\widehat{M}}$. Then \widehat{t} is also a left translation. Clearly $\widehat{t} \in \mathbb{N} = \mathcal{N}^G$.

Fact 3. *There exists a lift $\tilde{t}: N \rightarrow N$ of \widehat{t} such that $\tilde{t}a\tilde{t}^{-1} = a$ for all $a \in \pi_1(M)$.*

Proof. Consider the group $\widetilde{\mathcal{N}}^G$ of all lifts of members of \mathcal{N}^G to the universal cover N . Since $\mathcal{N}^G \cap G = 1$ the group \mathcal{N}^G acts effectively on M and the sequence

$$1 \rightarrow \pi_1(M) \rightarrow \widetilde{\mathcal{N}}^G \rightarrow \mathcal{N}^G \rightarrow 1$$

is exact. There is a homomorphism

$$H: \mathcal{N}^G \rightarrow \text{Out}(\pi_1(M))$$

induced by conjugation by elements of $\tilde{\mathcal{N}}^G$. (See Section IV.6 of [Br82].)

Take any lift $\bar{t} \in \tilde{N}^G$ of \hat{t} . Recall that t is homotopic to identity. This implies that $H(t) = [id_{\pi_1 M}]$. Therefore the corresponding automorphism of $\pi_1(M)$

$$a \mapsto \bar{t} a \bar{t}^{-1}$$

is an inner automorphism

$$a \mapsto b^{-1} a b$$

where $b \in \pi_1(M)$. Then $\tilde{t} \stackrel{\text{def}}{=} b\bar{t}$ is the posited lift. \square

Fact 4. *Assume that $f: \widehat{M} \rightarrow \widehat{M}$ is an affine diffeomorphism of the nilmanifold \widehat{M} homotopic to $id_{\widehat{M}}$. Then f is translation.*

Proof. Let $\tilde{f}: N \rightarrow N$ be a lift of f . Then \tilde{f} has the form $\tilde{f}(x) = vA(x)$, where $v \in N$ and A is an automorphism of N . Automorphism A restricts to an automorphism of Γ . And, since f is homotopic to identity, $A(\gamma) = a\gamma a^{-1}$, where $\gamma \in \Gamma$ and $a \in \Gamma$ is fixed. By [M49, Theorem 5] $A|_{\Gamma}$ extends uniquely to an automorphism of N . Hence

$$\forall x \in N \quad A(x) = axa^{-1}$$

and we see that $x \mapsto vax$ is a translation that covers f . \square

We start the proof of Proposition 7. Let x_0 be a fixed point of L . Denote by \hat{L} and \hat{x}_0 lifts of L and x_0 to \widehat{M} respectively. Also denote by \hat{h} a lift of h which is homotopic to $id_{\widehat{M}}$.

Notice that $h(x_0)$ is also fixed by L . Clearly \hat{x}_0 and $\hat{h}(\hat{x}_0)$ are periodic for \hat{L} and, therefore, are fixed by some power \hat{L}^q . Thus $\hat{h}(\hat{L}^q(\hat{x}_0)) = \hat{L}^q(\hat{h}(\hat{x}_0))$ for some $q > 0$, but $\hat{h} \circ \hat{L}^q = g_0 \circ \hat{L}^q \circ \hat{h}$ for some $g_0 \in G$. Hence, since G acts freely, $g_0 = id_{\widehat{M}}$. Therefore \hat{L}^q and \hat{h} commute and Theorem 2 of Walter's paper [W70] implies that \hat{h} is an affine diffeomorphism of \widehat{M} . By Fact 4 \hat{h} must be a translation; that is, $\hat{h} \in \mathcal{N}$.

Clearly \hat{h} normalizes G . Therefore, by Fact 2, $\hat{h} \in \mathcal{N}^G$ and, by Fact 3, \hat{h} admits a lift \tilde{h} in $N^{\pi_1 M}$. Fact 1 implies that there is a path that connects \tilde{h} to id_N in $N^{\pi_1 M}$. This path projects to a path that connects h and id_M . Thus h is isotopic to identity.

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